Morse-Radó theory for minimal surfaces in 3-manifolds.

Francisco Martín, UGR

UNIVERSIDAD DE GRANADA

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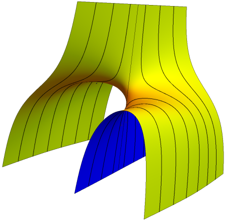


The intersection of two minimal surfaces

An interesting tool in the study of several problems in minimal surface theory consists of answering this question:

Question

How is the intersection of two minimal surfaces S_1 and S_2 ?

A seminal work on this type of problems was:

> **Tibor Rado**. On the Problem of Plateau. *Chelsea Publishing Co., New York, N. Y., 1951. (Photographic reproduction of a book which first appeared as Band 2, Heft 2 in the series Ergebnisse der Mathematik and ihrer Grenzgebiete, Springer, Berlin, 1933.)*

The intersection of two minimal surfaces

Theorem (Rado, 1933)

Let D_1 a bounded convex domain in the (x_1, x_2)-plane, and let $C_1 = \partial D_1$ be its boundary. Let $\varphi_k(x_1, x_2)$, $k = 3, \ldots, n$ be arbitrary continuous functions on C_1. Then there exists a solution

$$u(x_1, x_2) = (u_3(x_1, x_2), \ldots, u_n(x_1, x_2))$$

of the minimal surface equation (1) in D_1 with $u_k|C_1 = \varphi_k$, $k = 3, \ldots, n$.

$$\left(1 + \|u_{x_2}\|^2\right) u_{x_1 x_1} - 2\left(u_{x_1} \cdot u_{x_2}\right) u_{x_1 x_2} + \left(1 + \|u_{x_1}\|^2\right) u_{x_2 x_2} = 0. \qquad (1)$$

The intersection of two minimal surfaces

- If we are working in $\mathbb{R}^3$, and $p \in S_1 \cap S_2$ is a point so that the normal vectors are not parallel, then $S_1 \cap S_2$ is an analytic curve around p. We say that *p is a point of contact of order 0*.

If $\nu_1(p) = \pm\nu_2(p)$, then, up to a rigid motion that $p = (0,0,0)$ and $T_pS_1 = T_pS_2 \equiv \{z = 0\}$. If we write $S_1 = \mathrm{graph}(u_1)$ and $S_2 = \mathrm{graph}(u_2)$, where u_1 and u_2 are local solutions of

$$(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0 \tag{2}$$

If $u_1 \equiv u_2$, then (by analytic continuation) $S_1 \equiv S_2$.

The intersection of two minimal surfaces

If $u_1 \neq u_2$, then

$$(u_1 - u_2)(x, y) = \sum_{i \geq n} h^{(i)}(x, y).$$

where $h^{(i)}$ is an homogeneous harmonic polynomial of degree i. Then we say that S_1 and S_2 have **a contact of order $n-1$ at p**.

Theorem

If S_1 and S_2 have a contact of order $n-1$ at p, then $S_1 \cap S_2$ consists of n analytic curves meeting at p forming an angle of π/n.

An easy proof can be found in §437 of

Nitsche, Johannes C. C. Lectures on minimal surfaces. Vol. 1. *Cambridge University Press, Cambridge, 1989*

The intersection of two minimal surfaces

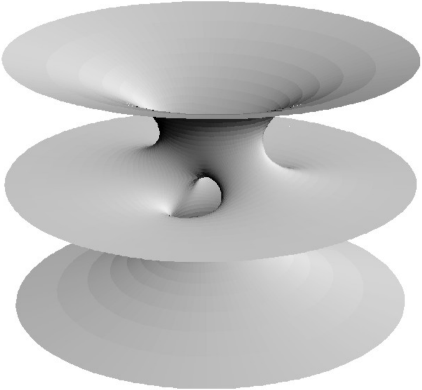


Figure: Costa minimal torus.

The intersection of two minimal surfaces

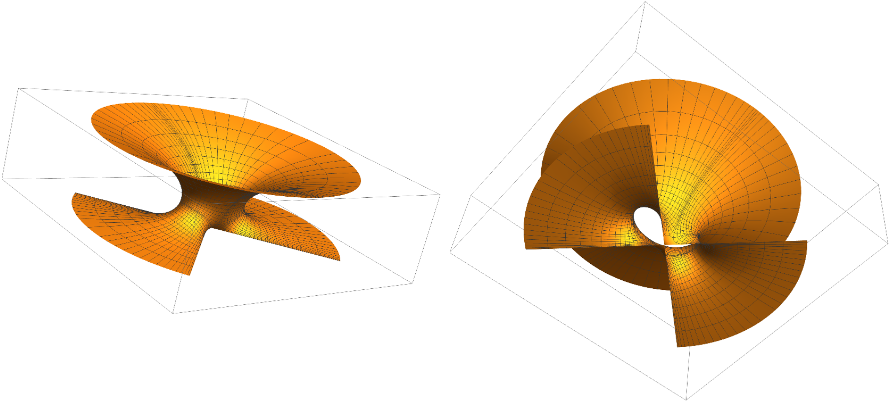


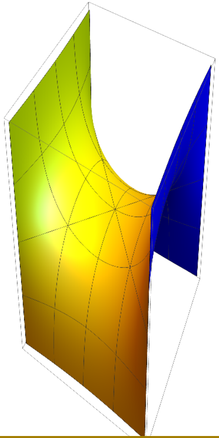
Figure: Costa minimal torus.

The intersection of two minimal surfaces

A nice application is an easy proof of this classical theorem:

Theorem (Bernstein, 1905)

Let S be a minimal surface which is an entire graph, then S is a plane.



Rado-type functions

Definition (Rado-type function)

A function $F : M \to \mathbb{R}$ on a surface M is called a **Rado-type function** if F is continuous on M and smooth on $M \setminus \partial M$, and if for each interior critical point p, either

1. $F|M$ is constant on a neighborhood of p, or
2. In suitable polar coordinates for M at p, F has the form

$$F(r, \theta) = F(p) + c\, r^n \sin n\, \theta + o(r^n) \tag{3}$$

   for some integer $n \geq 2$ and some $c \neq 0$.

The **multiplicity** of the critical point p is defined to be 0 in Case (1) and $(n-1)$ in Case (2).

Rado-type functions

The following theorem shows how Rado-type functions arise naturally.

Theorem (of Structure)

Let Ω be a smooth Riemannian 3-manifold and $F : \Omega \to \mathbb{R}$ be a smooth function with nowhere vanishing gradient such that the level sets of F are minimal surfaces. If M is a minimal surface in Ω, then $F|M$ is a Rado-type function.

In the previous setting, for S_1, S_2 minimal surfaces in $\mathbb{R}^3$

$$F(x, y, z) = z - u_2(x, y).$$

Then $F|S_1$ is a Rado-type function.

Proof of Structure Theorem

It suffices to consider **the case when $\Omega \subset \mathbb{R}^3$ with a metric g and** $p = (0, 0, 0)$.
We may assume that $F(0, 0, 0) = 0$. By applying a diffeomorphism to F and to the metric, we can assume that

$$F(x, y, z) = z.$$

Note that (near $(0, 0, 0)$), M is the graph of a function

$$u : \mathbf{B}^2(0, \varepsilon) \to \mathbb{R}$$

with

$$u(0, 0) = 0 \quad \text{and} \quad Du(0, 0) = \vec{0}.$$

Proof of Structure Theorem

Now u **and the** 0 **function are both solutions of the** g**-minimal surface equation**, so their difference $u = u - 0$ solves a second-order linear elliptic equation:

$$a_{ij} D_{ij} u + b_i D_i u + cu = 0 \qquad (4)$$

where a_{ij}, b_i, and c are smooth functions of (x, y).
(See, for example, **equation (10.22)** and the subsequent displayed formula in

> **Gilbarg, David; Trudinger, Neil S.** Elliptic partial differential equations of second order. Reprint of the 1998 edition. *Classics in Mathematics. Springer-Verlag, Berlin, 2001.*

.
There, the derivation is for solutions of differential inequalities, but the same proof works when equality holds.)

Proof of Structure Theorem

By making a linear change of coordinates, we can assume that

$$a_{ij}(0,0) = \delta_{ij}.$$

- If $u(x,y) = o(|(x,y)|^m)$ for all m, then u is identically 0 near the origin by standard unique continuation theorems. You can see for instance Theorem 1.1 and Corollary 1 in

  Micallef, Mario J.; White, Brian. The structure of branch points in minimal surfaces and in pseudoholomorphic curves. *Ann. of Math. (2) 141 (1995), no. 1, 35–85.*

- Otherwise, from the Taylor series for u, we see that

$$u(x,y) = h(x,y) + o(|(x,y)|^m)$$

for some $m \geq 2$ and some nonzero homogeneous polynomial h of degree m.

Proof of Structure Theorem

Thus

$$\begin{aligned} a_{ij}D_{ij}u + b_iD_iu + cu &= a_{ij}(0,0)D_{ij}u + (a_{ij} - a_{ij}(0,0))D_{ij}u + b_iD_iu + cu \\ &= \Delta u + (a_{ij} - a_{ij}(0,0))D_{ij}u + b_iD_iu + cu \\ &= \Delta h + o(r^{m-2}), \end{aligned} \tag{5}$$

where $r = |(x,y)|$. Now Δh is a **homogeneous polynomial of degree $m-2$**, so by (4) and (5),

$$\Delta h = 0.$$

We are done, because we can use (x,y) as local coordinates on M, and for $(x,y,z) \in M$,

$$F(x,y,z) = z = u(x,y).$$

Q.E.D.

Maximum principle

The following is an immediate consequence of the definition of Rado function:

Proposition (Maximum principle)

If a Rado-type function F on a connected surface is non-constant, then its critical points are isolated, and F has no local maxima or minima.

Corollary

If F is a Rado-type function on M, and if J is a point or interval (possible infinite) in $\mathbb{R}$, then

$$d_1(M \cap F^{-1}(J)) \leq d_1(M),$$

where $d_i(\Sigma) := \dim H_i(\Sigma, \mathbb{Z}_2)$.

Maximum principle

Proof.

Let C be a closed curve, not necessarily connected in $M \cap F^{-1}(J)$, that is homologically trivial (mod 2) in M.
Then C bounds a region R in M. The maximum and minimum of F on R must occur on C, by the previous proposition.
Thus, R lies in $M \cap F^{-1}(J)$ and therefore C is homologically trivial in $M \cap F^{-1}(J)$.
We have shown that the inclusion of $M \cap F^{-1}(J)$ into M induces a monomorphism of first homology. □

Level curves of a Rado function

Theorem (Level curves)

Suppose that M is a surface with boundary and that $F : M \to \mathbb{R}$ is a Rado-type function that is smooth on $M \setminus \partial M$. Let $t \in \mathbb{R}$ and G be $M \cap \{F = t\}$. Suppose that G is compact and contains no closed loops and that the number k of points in $(\partial M) \cap \{F = t\}$ is finite. Then G is a finite union of finite trees, and

$$2(c + m) \leq k$$

where c is the number of connected components of G and m is the sum of the multiplicities of the critical points of F on $\{F = t\}$.

Level curves of a Rado function

Proof.

Let T be a **connected component of** $G \setminus \partial M$. Then T is a tree, and no two ends of T go to the same point in ∂M. The number of ends of T is $2(m(T) + 1)$, where $m(T)$ **is the sum of the multiplicities of the critical points in** T.

If there were infinitely many such trees, then there would be two of them T and T' with exactly the same endpoints on ∂M. But then $T \cup T'$ would contain a loop, **a contradiction**.

Thus G is a finite tree. As before, the number of ends is $\geq m + c$ and is less than or equal to the number of points in $(\partial M) \cap \{F = t\}$. □

This theorem has an interesting and useful corollary.

Level curves of a Rado function

Corollary

*Suppose in the previous theorem that $d_1(M) < \infty$. Let $G = M \cap \{F = t\}$. Then (**whether or not G contains loops**) G is a finite network,*

$$d_1(G) \leq d_1(M),$$

and

$$2(c + m) \leq k + 2d_1(G).$$

Let S be the set consisting of interior critical points of F that lie on G together with the points of $(\partial M) \cap \{F = t\}$. The statement that G is a finite network is the statement that S is finite and the $G \setminus S$ has finitely many connected components. The components of $G \setminus S$ are called edges of G.

Level curves of a Rado function

Proof.

By the corollary to the maximum principle, we know that $d_1(G) \leq d_1(M)$. We prove this corollary by induction on $d_1(G)$. If $d_1(G) = 0$, this is the previous theorem.

Thus suppose $d_1(G) > 0$. Then G contains a closed loop L. Let E be one of the edges of G that lies in L, let p be a point in the E, and let M' and G' be obtained from M and G by removing a small open ball around p. Then

$$d_0(G') = d_0(G),$$
$$d_1(G') = d_1(G) - 1,$$

and $(\partial M') \cap \{F = t\}$ has exactly $k + 2$ points. Thus by induction, G' is a finite network, and therefore G is a finite network. Also, by induction,

$$2(m + d_0(G')) \leq k + 2 + 2d_1(G') \Rightarrow 2(m + d_0(G)) \leq k + 2d_1(G)$$

Theorem (Rado, 1933)

Let D_1 a bounded convex domain in the (x_1, x_2)-plane, and let $C_1 = \partial D_1$ be its boundary. Let $\varphi_k(x_1, x_2)$, $k = 3, \ldots, n$ be arbitrary continuous functions on C_1. Then there exists a solution

$$u(x_1, x_2) = (u_3(x_1, x_2), \ldots, u_n(x_1, x_2))$$

of the minimal surface equation (1) in D_1 with $u_k | C_1 = \varphi_k$, $k = 3, \ldots, n$.

Proof

The functions $\varphi_k(x_1, x_2)$ define a Jordan curve Γ in $\mathbb{R}^n$ which projects onto C_1.
By Douglas' theorem there exists a continuous map

$$x(u_1, u_2) : \overline{D} \to \mathbb{R}^n$$

where $\overline{D} = \{\|(u_1, u_2)\| \leq 1\}$, which defines a minimal surface in D and takes $C = \partial D$ homeomorphically onto Γ.
The the map

$$\psi(u_1, u_2) := (x_1(u_1, u_2), x_1(u_1, u_2))$$

is a continuous map of $\overline{D}$ which is harmonic in D and maps C homeomorphically onto C_1. By the convex hull property for minimal surfaces

$$\psi(D) = D_1$$

Proof

Moreover $\mathrm{Jac}(\psi)$ **never vanishes** in D. Suppose not, then there exists a point $p \in D$ with $\mathrm{Jac}(\psi)(p) = 0$. This means that there exists $a, b \in \mathbb{R}$ such that

$$aDx_1(p) + bDx_2(p) = \vec{0}.$$

Define $F := a\,x_1 + b\,x_2$, it is a Rado type function and $G = \{F = F(p)\}$ doesn't contain any loop. Then

$$2(m + c) \leq k.$$

But $2(m + c) \geq 4$ and $k \leq 2$, a contradiction. Then ψ a local diffeomorphism. As $\psi|C$ is 1-to-1 onto C_1, then ψ is a diffeomorphism. So,

$$u_k := x_k \circ \psi^{-1}, \quad k = 3, \ldots, n.$$

Q.E.D.

Rado-type theorems

Theorem (Rado Alt)

Suppose that M is homeomorphic to a **compact manifold-with-boundary***, that each component of M has nonempty boundary, and that $F : M \to \mathbb{R}$ is a Rado-type function. Suppose F is not constant on any component of ∂M, and that $F|\partial M$ has $n(F) < \infty$ local minima (in the sense described below). Then*

$$N(F) \leq n(F) - \chi(M)$$

$\chi(M) \equiv$ *Euler characteristic of M*
$N(F) \equiv$ *number of critical of F in M (counted with multiplicity.)*

Definition

Here, by a **"local minimum"** of $F|\partial M$, we mean a set K consisting of **a point or a closed arc** such that for some open arc $U \subset \partial M$ containing K, F takes a constant value c on K and $F > c$ at all points of $U \setminus K$.

Proof of the theorem Rado Alt

Without loss of generality, we can assume that:

- $(\partial M) \cap \{F = t\}$ is finite for every t.

If not, we define an equivalence relation on ∂M as follows:

equivalence relation

$x \sim y$ if F is constant on a connected subset of ∂M containing x and y.

Instead of working with M, we can work with $M/\sim$.

By passing to a double cover, the result for **non-orientable M** follows from the result for orientable M.

- Thus **we assume M is orientable**.

Proof of the theorem Rado Alt

By elementary topology, the number of boundary components of M is greater than or equal to the Euler characteristic, with equality if only if M is a union of disks.

Since $F|\partial M$ has to have a local minimum on each component of ∂M, we see that

$$n(F) - \chi(M) \geq 0,$$

with **equality if and only M is a union of disks and $F|\partial M$ has exactly one local minimum in each component of ∂M**.

In this case, F has no critical points by the theorem about the level curves.

As F has only one minimum on each connected component of ∂M then each component of $\{F = t\}$ meets ∂M in two points (otherwise there would be more than 2 local minima.)

Then the theorem says $2(c + m) \leq k = 2c \Rightarrow m = 0$, and this happens for every t.

Proof of the theorem Rado Alt

We prove the result by induction on $n(F) - \chi(M)$.
(It might seem more natural to do induction on the number $N(F)$ of critical points. However, we do not know ahead of time that the number is finite.)
Since $n(F) - \chi(M) \geq 0$, we are done if M has no critical points.

Proof of the theorem Rado Alt

- Thus suppose $F|M$ has an interior critical point p.

Let K be a compact subset of the interior of M such that p is in the interior of K and such that K does not contain any other critical points of F.
Let $u : M \to \mathbb{R}$ be a smooth function that is **supported in K** and that is equal to 1 in a neighborhood of U. For all t sufficiently close to 0, $F + tu$ will also be Rado-type function. Thus (by replacing F by a suitable $F + tu$) we may assume that the level set of p contains no other critical points, and that $M \cap \{F = F(p)|\}$ does not contain any other points in ∂M. Recall that, by first sentence of proof, $M \cap \{F = F(p)\}$ does not contain any arcs of ∂M.

Proof of the theorem Rado Alt

Let m **be the multiplicity of the critical point** p. If U is a little geodesic neighborhood of p, then $U \cap \{F \neq F(p)\}$ is the union of $2(m+1)$ wedges that meet at p. In half of the wedges, $F > F(p)$ and in the other half, $F < F(p)$.

In each wedge, we can take a little arc C with one endpoint at p such that F is strictly monotonic along C. Since F is has no internal local maximum/minimum $\Longrightarrow$ we can extend the arc, keeping $F|C$ strictly monotonic along it, until we reach a point in ∂M.

(For example, we can follow the gradient flow or minus the gradient flow, perturbing slightly, if needed, to avoid critical points.)

Thus we get an arc with one endpoint at p and the other endpoint q in ∂M.

We can choose the arcs so that no two of them meet at the same point in ∂M, and so that none of the points are local maxima or local minima of $F|\partial M$.

Proof of the theorem Rado Alt

- Let T be the union of those $2(m+1)$ arcs.

Cut M along T to get a new manifold-with-boundary M'. Note that p becomes $2(m+1)$ points in $\partial M'$. **None of those points is a local minimum of $F|\partial M'$.**

We wish to compare the Euler characteristics of M and of M'. For this, it is useful to consider a triangulation of M whose edges and vertices include the edges of vertices of T. We use the corresponding triangulation of M'.

Proof of the theorem Rado Alt

Now each arc C in T from p to a point $q \in \partial M$ becomes two arcs in M', and the end point q becomes two points in M'.
Thus the change in Euler characteristic caused by replacing C by two arcs cancels the change caused by replacing q by two points.
Hence the change in Euler characteristic comes from replacing p in M by $2(m+1)$ points in M':

$$\chi(M') = \chi(M) - 1 + 2(m+1) = \chi(M) + 2m + 1. \tag{6}$$

Proof of the theorem Rado Alt

If $F(q) > F(p)$, then there are no local minima of $F|\partial M'$ on the two arcs corresponding to C. However, if $F(q) < F(p)$, then q will correspond to two points in $\partial M'$, and exactly one will be a local minimum of $F|\partial M'$. Thus

$$n(F|M') = n(F|M) + m + 1. \tag{7}$$

By (6) and (7),

$$n(F|M') - \chi(M') = n(F|M) - \chi(M) - m. \tag{8}$$

Proof of the theorem Rado Alt

In particular, $n(F|M') - \chi(M') < n(F|M) - \chi(M)$, so by induction (recall we are doing induction on $n - \chi$) we can assume that the theorem is true for M':

$$N(F|M') \leq n(F|M') - \chi(M'). \tag{9}$$

Also,

$$N(F) = N(F|M') + m. \tag{10}$$

Combining (8), (9), and (10), we see that $N(F) \leq n(F|M) - \chi(M)$.

Q.E.D

Sharper Rado's estimate

We can get a sharper estimate as follows.

Theorem (Sharper Rado's estimate)

Let F and M be as above. Let $\sigma(F)$ **be the number of local maxima of** $F|\partial M$ **that are not local maxima of** F **plus the number of local minimal of** $F|\partial M$ **that are not local minima of** F. *Then*

$$N(F) + \sigma(F) \leq n(F) - \chi(M). \tag{11}$$

Equivalently,

$$N(F) \leq c_0(F) - c_1(F) - \chi(M),$$

where $c_0(F)$ *is the number of local minima of* $F|\partial M$ *that are also local minima of* F, *and* $c_1(F)$ *is the number of local maxima of* $F|\partial M$ *that are not local maxima of* $F|M$.

Proof of sharper Rado's estimate

- As in the proof of Theorem 5, we can assume that F is not constant along any arc of ∂M.

Note that the number of local maxima of $F|\partial M$ is equal to the number of local minima, from which it follows that

$$\sigma(F) \leq 2n(F).$$

Thus $\sigma(F)$ is finite. **We prove (11) by induction on $\sigma(F)$**.

- If $\sigma(F) = 0$, this is Theorem 5.
- Thus suppose $\sigma(F) > 0$. The there is a point $p \in \partial M$ such that

1. p is a local minimum of $F|\partial M$ but not of F, or
2. p is a local maximum of $F|\partial M$ but not of F.

Proof of sharper Rado's estimate

- **Case 1: p is a local minimum of $F|\partial M$ but not of F** Let C be an arc in M with one end point at p and such that F is strictly decreasing as we move C away from p. We can extend C (keeping $F|C$ strictly monotonic) until it reaches a point q of ∂M. We may choose C so that q is neither a local maximum nor a local minimum of $F|\partial M$.
- **Now cut M along C to get M'.** Note that neither of the two points in $\partial M'$ corresponding to p is a local minimum or a local maximum of $F|\partial M'$. On the other hand, one of the two points, call it $\tilde{q}$, corresponding to q is a local minimum of $F|M'$.

Proof of sharper Rado's estimate

Thus

$$n(F|M') = n(F) \tag{12}$$

(because we lost the local minimum p of $F|\partial M$ but we also gained the local minimum $\tilde{q}$ of $F|\partial M'$) and

$$\sigma(F|M') = \sigma(F) - 1 \tag{13}$$

(because we lost one σ-type point, **namely** p, and we did not gain any.) Also

$$\begin{aligned} N(F|M') &= N(F), \\ \chi(M') &= \chi(M) + 1. \end{aligned} \tag{14}$$

Proof of sharper Rado's estimate

By (12), (13), (14), and induction,

$$\begin{aligned} N(F) + \sigma(F) &= N(F|M') + \sigma(F|M') + 1 \\ &\leq n(F|M') - \chi(M') + 1 \\ &= n(F) - \chi(M). \end{aligned}$$

This completes the proof in Case 1 .

Proof of sharper Rado's estimate

- Case 2: p is a local maximum of $F|\partial M$ but not of F. Let C be an arc from p to a point $q \in \partial M$ such that $F|C$ is strictly increasing as we move away from p.
We choose C so that q is not a critical point of $F|\partial M$.
- As in Case 1, $n(F|M') = n(F)$ (though the reason is slightly different: in this case, none of the points of C are local minima of $F|\partial M$, and none of the corresponding points in M' are local minima of $F|\partial M'$.)
Also, as in Case 1, $\sigma(F|M') = \sigma(F) - 1$. The rest of the proof is exactly as in Case 1.

Q.E.D.

Rado type functions over non-compact surfaces

In some situations, the Rado-type function F is defined only on a subset of a surface M, and that subset may not be compact. With some mild addition hypotheses, the following theorem allows us to estimate the number of critical points of F.

Theorem

Suppose M is a 2-manifold with boundary and that $\{F \leq r\}$ is compact for every r. Suppose that:

1. *$F|\partial M$ has only finitely many local minima.*
2. *$H_1(M, \mathbb{Z}_2)$ is finite.*
3. *Outside of a compact subset J of ∂M, M is a smooth manifold with boundary, F is smooth on $M \setminus J$, and $F|\partial M$ has no critical points in $(\partial M) \setminus J$.*

Then

$$N(F) \leq n(F) - \sigma(F) - \chi(M).$$

Proof.

Let $r > \max(F|J)$ and let

$$M_r = M \cap \{F \leq r\}.$$

Note that

$$n(F|M_r) = n(F),$$
$$\sigma(F|M_r) = \sigma(F).$$

(The components of $M \cap \{F = r\}$ are local maxima of $F|M_r$ and hence do not get counted in $n(F|M_r)$ or $\sigma(F|M_r)$.)
Recall that $d_i(\Sigma)$ denotes the dimension of $H_i(\Sigma, Z_2)$.
Then

$$\begin{aligned} N(F|M_r) &\leq n(F|M_r) - \sigma(F|M_r) - \chi(M_r) \\ &= n(F) - \sigma(F) - \chi(M_r). \end{aligned}$$

Proof.

Since $\chi(M_r)$ is the alternative sum of $d_i(M_r) := \dim H_i(M_r, Z_2)$,

$$N(F|M_r) \leq n(F) - \sigma(F) + d_1(M_r).$$

Now $d_1(M_r) \leq d_1(M)$ since $H_1(M_r, Z_2)$ injects into $H_1(M, Z_2)$. Thus

$$N(F|M_r) \leq n(F) - \sigma(F) + d_1(M).$$

Letting $r \to \infty$ shows that $N(F|M_r)$ is finite. Now choose r greater than $\max(F|J)$ and greater than all the critical values of F. Then $N(F) = N(F|M_r)$. Since DF and $D(F|\partial M)$ do not vanish outside of M_r, M is homotopy equivalent to M_r and thus $\chi(M_r) = \chi(M)$. Thus

$$N(F) \leq n(F) - \sigma(F) - \chi(M).$$

□

The Smooth Case

Here we show that in many situations, equality holds:

Theorem (Smooth I)

Suppose M is a smooth, compact surface-with-boundary, that F is a smooth Rado function, that $F|\partial M$ is a Morse function, and that DF never vanishes on ∂M.
Then

$$N(F) = c_0(F) - c_1(F) - \chi(M). \tag{15}$$

Recall that:

- $c_0(F)$ is the number of local minima of $F|\partial M$ that are also local minima of F, and
- $c_1(F)$ is the number of local maxima of $F|\partial M$ that are not local maxima of $F|M$.

The smooth case

Theorem (Smooth II)

Suppose $a < b$ are regular values of $F|M$ and of $F|\partial M$ and let

$$M[a,b] = M \cap \{a \leq F \leq b\},$$
$$M[a] = M \cap \{F = a\}.$$

Let $c_i(M[a,b])$ be the number of c_i-type points of M that lie in $a \leq F \leq b$ (or, equivalently, that lie in $a < F < b$ since a and b are regular values of $F|\partial M$.) Then

$$N(F|M[a,b]) = c_0(M[a,b]) - c_1(M[a,b]) - \chi(M[a,b]) + \chi(M[a]). \quad (16)$$

Furthermore, for (16) to hold, M **need not be compact***;* **it suffices for $M[a,b]$ to be compact***.*

Proof of Smooth II

It suffices to prove Smooth II, since if M is compact, then (15) follows by letting $a < \min F$ and $\max F < b$.

Claim

The formula (16) *holds when M is compact and F has no critical points.*

Proof of claim.

The result is standard elementary Morse Theory. When $b > a$ is close to a, then there are no critical points of $F|\partial M$ in $F^{-1}[a, b]$, so $M[a, b]$ is homeomorphic to $M[a] \times [a, b]$, and therefore

$$\chi(M[a, b]) = \chi(M[a]).$$

Also, $c_0(F|M[a, b]) = c_1(F|M[a, b]) = 0$. Thus (16) holds when $b > a$ is close to a.

As we increase b, the Euler characteristic of $M[a, b]$ increases by 1 each time we pass one of the c_0-type critical points (a new connected component is created) and it decreases by 1 each time we pass one of the c_1-type critical points. This completes the proof of Claim 1. □

Proof of Smooth II

Note that whether or not M is compact, we can find a compact region $M^* \subset M$ with smooth boundary such that $M^*[a,b] = M[a,b]$, such that $M^* \setminus M[a,b]$ contains no critical points of F, and such that $F|\partial M^*$ is a Morse Function. None of the terms in (16) change if we replace M by M^*. Thus it suffices to prove (16) assuming that M is compact and that all the critical points of M lie in $M[a,b]$.

Proof of Smooth II

We prove (16) by induction on the number of critical points of F.
If F has no critical points, then the formula holds by Claim 1.
Now suppose there is a critical point p of F. Since a and b are regular values, p is in the interior of $M[a, b]$. In suitable local coordinates,

$$F(r, \theta) = F(p) + r^n \sin n\,\theta + o(r^n)$$

in a neighborhood of the point, where $n - 1$ is the multiplicity of the critical point.
Now let M' be obtained from M by removing a very small open disk $\{r < \varepsilon\}$ about p. We choose ε small enough that the closure of the disk is contained in the interior of $M[a, b]$.
Thus

$$\chi(M'[a, b]) = \chi(M[a, b]) - 1.$$

Let Γ be the boundary of that disk.

Proof of Smooth II

Then $F|\Gamma$ is a Morse function with n local maxima and n local minima. None are critical points of $F|M$. The local minima of $F|\Gamma$ are not local minima of $F|M'$, and the local maxima of $F|\Gamma$ are not local maxima of $F|M'$. Thus

$$\begin{aligned} c_0(F|M'[a,b]) &= c_0(F|M[a,b]), \\ c_1(F|M'[a,b]) &= c_1(M[a,b]) + n. \end{aligned}$$

By induction, the counting formula (15) is true for $M'[a,b]$. Thus

$$\begin{aligned} \mathrm{N}(F|M[a,b]) &= \mathrm{N}(F|M'[a,b]) + (n-1) \\ &= c_0(F|M'[a,b]) - c_1(F|M'[a,b]) \\ &\qquad - \chi(M'[a,b]) + \chi(M'[a]) + (n-1) \\ &= c_0(F|M[a,b]) - (c_1(F|M[a,b]) + n) \\ &\qquad - (\chi(M[a,b]]) - 1) + \chi(M[a]) + (n-1) \\ &= c_0(F|M[a,b]) - c_1(F|M[a,b]) - \chi(M[a,b]) + \chi(M[a]). \end{aligned}$$

Applications

Definition

We say that $M \subset \mathbb{R}^3$ is a translator with **velocity** v if

$$M \mapsto M + t\, v$$

is a **mean curvature flow**.

Remark

This is equivalent to say

$$\vec{H} = v^{\perp}.$$

Up to a rigid motion and a homothety we can assume that $v = (0, 0, -1)$. Then the translator equation has the form

$$\vec{H} = (0, 0, -1)^{\perp}.$$

Translators as minimal surfaces

In 1994, T. Ilmanen observed that M is a translator iff M is minimal with respect the metric

$$g_{ij} := \mathrm{e}^{-z}\delta_{ij}.$$

This allows us to use:

1. compactness theorems,
2. curvature estimates,
3. maximum principles,
4. monotonicity,

for g-minimal surfaces. Moreover, reflection in vertical planes and 180^o-rotation about vertical lines are isometries of g. Therefore, we can use **Schwarz reflection** and **Alexandrov method of moving planes** in our context.

Translating graphs

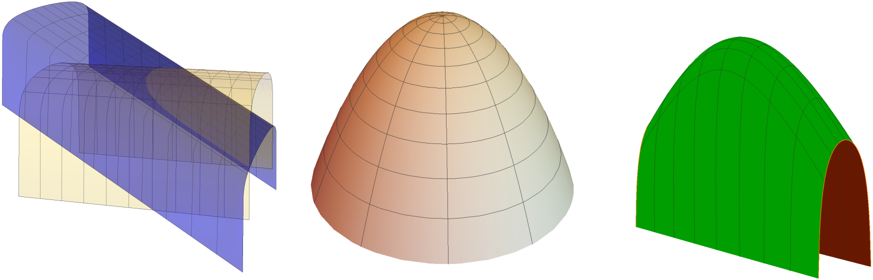


Figure: Some examples of complete graphical translators.

Theorem (Classification Theorem, Hoffman-Ilmanen-M-White)

For every $w > \pi$, there exists (up to translation) a unique complete translator $u : \mathbb{R} \times (0, w) \to \mathbb{R}$. for which the Gauss curvature is everywhere > 0. The function u is symmetric with respect to $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, w - y)$ and thus attains its maximum at $(0, w/2)$. Up to isometries of $\mathbb{R}^2$ and vertical translation, the only other complete translating graphs are the tilted grim reapers and the bowl soliton, a strictly convex, rotationally symmetric graph of an entire function.

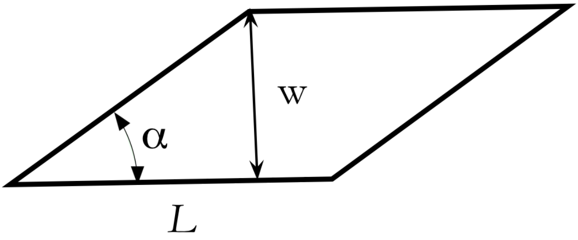
In particular (as Spruck and Xiao had already shown), there are no complete graphical translators defined over strips of width less than π. Moreover the grim reaper surface is the only example with width π.
The positively curved translator in the Classification Theorem is called a **Δ-wing**.

Definition

For $\alpha \in (0, \pi)$, $w \in (0, \infty)$, and $0 < L \leq \infty$, let $P(\alpha, w, L)$ be the set of points (x, y) in the strip $\mathbb{R} \times (0, w)$ such that

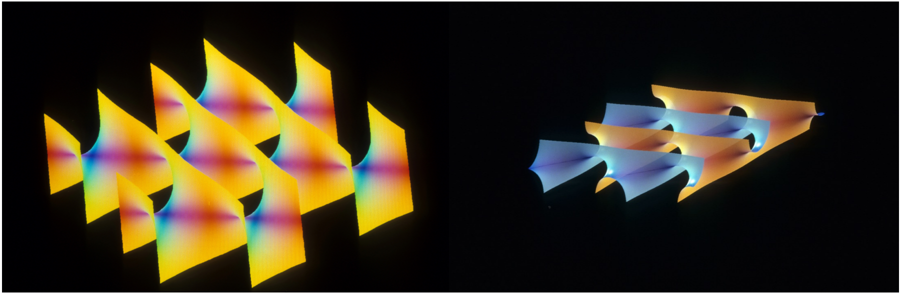
$$\frac{y}{\tan \alpha} < x < L + \frac{y}{\tan \alpha}.$$

The lower-left corner of the region is at the origin and the interior angle at that corner is α.



Classical Scherk's surfaces are obtained by solving this boundary problem:

$$(\circledast)\begin{cases} u : P = P(\alpha, w, L) \to \mathbb{R}, \\ \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \\ u = -\infty \text{ on the horizontal sides of } P, \\ u = +\infty \text{ on the nonhorizontal sides of } P \end{cases}$$



We are interested in solving this boundary problem:

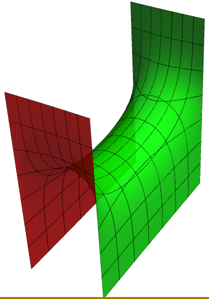
$$(\divideontimes\divideontimes)\begin{cases} u : P = P(\alpha, w, L) \to \mathbb{R}, \\ \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -\frac{1}{\sqrt{1+|\nabla u|^2}}. \\ u = -\infty \text{ on the horizontal sides of } P, \\ u = +\infty \text{ on the nonhorizontal sides of } P \end{cases}$$

Scherk translators

Theorem

For every $0 < \alpha < \pi$ and $0 < w < \pi$ there exists a unique $L = L(\alpha, w) > 0$ with the following property: there is a smooth surface-with-boundary $\mathcal{D} = \mathcal{D}_{\alpha,w}$ such that:

- $\mathcal{D}$ is a translator and $\mathcal{D} - \partial\mathcal{D}$ is the graph of a solution of (⋇⋇),
- Given $(\alpha, w, L(\alpha, w))$, the surface $\mathcal{D}$ is unique up to vertical translations,
- If $w \geq \pi$ and $0 < L < \infty$, the problem (⋇⋇) has no solutions.



Remark

- Since $P(\pi - \alpha, w, L)$ is the image of $P(\alpha, w, L)$ under reflection in the line $x = L/2$, it follows that $L(\alpha, w) = L(\pi - \alpha, w)$ and that $\mathcal{D}_{\pi-\alpha,w}$ is the image of $\mathcal{D}_{\alpha,w}$ under reflection in the plane $x = L(\alpha, w)/2$, followed by a vertical translation.
- $\mathcal{D}_{\alpha,w}$ has negative Gauss curvature everywhere,
- the Gauss map is a **diffeomorphism** from $\mathcal{D}_{\alpha,w}$ onto

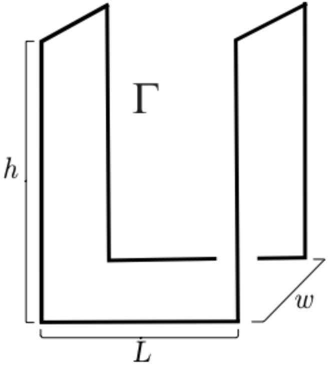
$$\mathbb{S}^{2+} \setminus Q$$

where $\mathbb{S}^{2+}$ is the closed upper hemisphere and where Q is the set consisting of the four unit vectors in the equator $\partial\mathbb{S}^{2+}$ that are perpendicular to the edges of the parallelogram $P(\alpha, w, L(\alpha, w))$ over which $\mathcal{D}_{\alpha,w}$ is a graph.

Proof of Existence

From now on, we are going to fix $0 < w < \pi$ and $0 < \alpha < \pi$.

We consider the polygonal Jordan curve Γ in $\mathbb{R}^3$



We can solve the corresponding **Plateau problem associated to Γ** (in $\mathbb{R}^3$ endowed with Ilmanen's metric) and so we get a minimal disk that we are going to denote $\mathcal{D}(L, h)$.

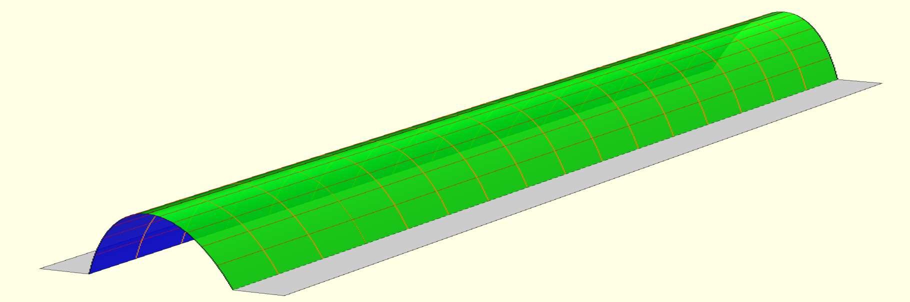
It has the following properties:

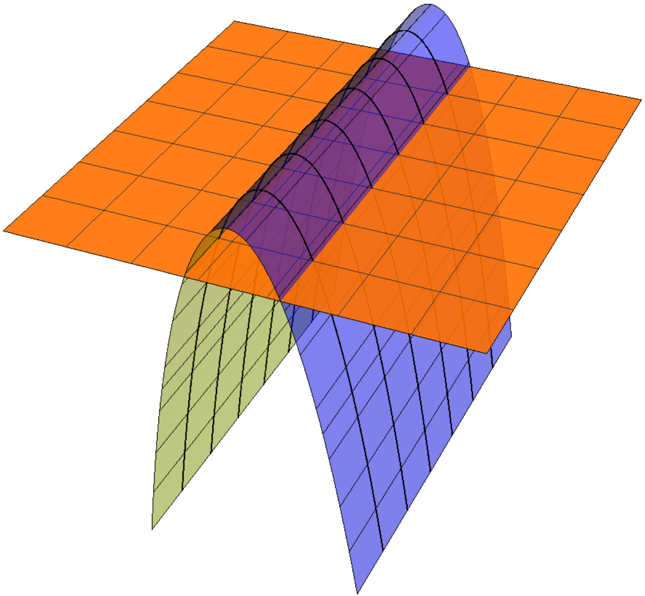
1. $\mathcal{D}(L, h)$ is a graph over the plane $z = 0$.
2. $\mathcal{D}(L, h)$ has the same symmetry as Γ.

For a fixed height h, we have that:

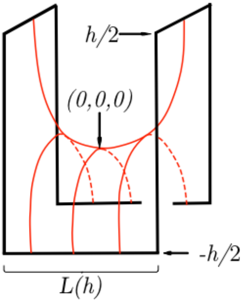
(i) As $L \to 0$, the limit of $\mathcal{D}(w, L, h)$ is the "U-shaped" curve described by Γ, as $L \to 0$.

(ii) As $L \to \infty$, the limit of $\mathcal{D}(w, L, h)$ is the part of the standard grim reaper cylinder which is a graph over the strip $\mathbb{R} \times (0, w)$.





This means that (for h sufficiently large) it is possible to find $L(h) \in (0, +\infty)$, such that the height of the saddle point (which is at the intersection of the symmetry line) is precisely $h/2$. Up to a translation in space we can assume that the saddle point is placed in the origin $(0, 0, 0)$.



Now, we would like to take limit of these family of disks $\{\mathcal{D}(L(h), h) \ : \ h > 0\}$, as the height $h \to \infty$.

Claim

The length $L(h)$ is bounded, as $h \to \infty$.

Again we proceed by contradiction. Assume that $\limsup L(h) = +\infty$.
Then we have a sequence $\{h_i\} \nearrow +\infty$ such that $\{L(h_i)\} \nearrow +\infty$.
We consider M the sub-sequential limit of $\mathcal{D}(L(h_i), h_i)$. M is a complete translator which is a graph over the strip $\mathbb{R} \times (-w/2, w/2)$ and $w < \pi$, which is impossible **(classification theorem)**.
This contradiction proves that $L(h)$ is bounded.
Then, given a sequence $\{h_i\} \nearrow +\infty$ so that $\{L(h_i)\} \nearrow L$, the limit $L < +\infty$. The limit of the sequence $\mathcal{D}(L(h_i), h_i)$ is the translator that we are looking for.

Proof of uniqueness

First step

Let P be the interior of a parallelogram in $\mathbb{R}^2$ with two sides parallel to the x-axis. Suppose

$$u, v : P \to \mathbb{R}$$

are translators that have boundary values $-\infty$ on the horizontal sides of P and $+\infty$ on the other sides. Then $u - v$ is constant.

If ξ is a vector in $\mathbb{R}^2$, let

$$u_\xi : P_\xi \to \mathbb{R}$$

be the result of translating $u : P \to \mathbb{R}$ by ξ, and let

$$\phi_\xi : P \cap P_\xi \to \mathbb{R},$$
$$\phi_\xi = u_\xi - v.$$

Suppose contrary to the theorem that $u - v$ is not constant. Then there is a point p where $Du(p) \neq Dv(p)$. We know that the image of the Gauss map of the graph of u is the entire upper hemisphere.

- Equivalently, $Du(\cdot)$ takes every possible value. Thus there is a $q \in P$ such that $Du(q) = Dv(p)$. Let $\xi_0 = p - q$. Then p is a critical point of ϕ_{ξ_0}.
- We claim that it is an isolated critical point. For otherwise ϕ_{ξ_0} would be constant near p and therefore (by unique continuation) constant throughout $P \cap P_{\xi_0}$, which is impossible. (Note for example that ϕ_{ξ_0} is $+\infty$ on some edges of $P \cap P_{\xi_0}$ and $-\infty$ on other edges.)

- It follows (Morse theory) that for every ξ sufficiently close to ξ_0, ϕ_ξ has a critical point. In particular, there is such a ξ for which P_ξ is in general position with respect to P.

- However, for such a ξ, ϕ_ξ cannot have a critical point, a contradiction. It cannot have a critical point because ϕ_ξ is $+\infty$ on two adjacent sides of its parallelogram domain $P \cap P_\xi$ and $-\infty$ on the other two sides.

(Note that P contains exactly one vertex of P_ξ, P_ξ contains exactly one vertex of P, and at each of the two points where an edge of P intersects an edge of P_ξ, one of the functions u and u_ξ is $+\infty$ and the other is $-\infty$.)

Second step

Suppose for $i = 1, 2$ that

$$u_i : P(\alpha, w_i, L_i) \to \mathbb{R}$$

is a graphical translator such that u_i has boundary value $-\infty$ on the horizontal edges of its domain and $+\infty$ on the nonhorizontal edges. Suppose also that $w_1 \leq w_2$ and that $L_1 \geq L_2$. Then $w_1 = w_2$, $L_1 = L_2$, and $u_1 - u_2$ is constant.

- The proof is almost identical to the proof of Step 1.

Non-existence of Scherk translators for $w \geq \pi$.

Proposition (Non-existence)

Let $w \geq \pi$ and $L < \infty$. Then there is no translator

$$u : P = P(\alpha, w, L) \to \mathbb{R}$$

with boundary values $-\infty$ on the horizontal sides and $+\infty$ on the nonhorizontal sides.